LCNS, Vol 1449, pp. 329–338, Springer 1998

On Multi-Dimensional Hilbert Indexings

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Abstract. Indexing schemes for grids based on space-filling curves (e.g., Hilbert indexings) find applications in numerous fields. Hilbert curves yield the most simple and popular scheme. We extend the concept of curves with Hilbert property to arbitrary dimensions and present first results concerning their structural analysis that also simplify their applicability. As we show, Hilbert indexings can be completely described and analyzed by "generating elements of order 1", thus, in comparison with previous work, reducing their structural complexity decisively.

1 Introduction

Discrete multi-dimensional spaces are of increasing importance. They appear in various settings such as combinatorial optimization, parallel processing, image processing, geographic information systems, data base systems, and data structures. In many applications it is necessary to number the points of a discrete multi-dimensional space (or, equivalently, a grid) by an indexing scheme mapping each point bijectively to a natural number in the range between 1 and the total number of points in the space. Often it is desirable that this indexing scheme preserves some kind of locality, that is, close-by points in the space are mapped to close-by numbers or vice versa. For this purpose, indexing schemes based on space-filling curves have shown to be of high value [4–9].

In this paper we study Hilbert indexings, perhaps the most popular space-filling indexing schemes. Properties of 2D and 3D Hilbert indexings have been extensively studied recently [4–10]. However, most of the work so far has focused on empirical studies. Up to now, little attention has been paid to the theoretical study of structural properties of multi-dimensional Hilbert curves, the focus of this paper. Whereas with "modulo symmetry" there is only one 2D Hilbert curve, there are many possibilities to define Hilbert curves in the 3D setting [4, 9]. The advantage of Hilbert curves is their (compared to other curves) simple structure.

Our results can shortly be sketched as follows. We generalize the notion of Hilbert indexings to arbitrary dimensions. We clarify the concept of Hilbert curves in multi-dimensional spaces by providing a natural and simple mathematical formalism that allows combinatorial studies of multi-dimensional Hilbert

* Work partially supported by a Feodor Lynen fellowship of the Alexander von Humboldt-Stiftung, Bonn, and the Center for Discrete Mathematics, Theoretical Computer Science, and Applications (DIMATIA), Prague.

indexings. For reasons of (geometrical) clearness, we base our formalism on permutations instead of e.g. matrices or other formalisms [2–4, 10]. So we obtain the following insight: Space-filling curves with Hilbert property can be completely described by simple generating elements and permutations operating on them. Structural questions for Hilbert curves in arbitrary dimensions can be decided by reducing them to basic generating elements. Putting it in catchy terms, one might say that for Hilbert indexings what holds "in the large" (i.e., for large side-length), can already be detected "in the small" (i.e., for side-length 2). In particular, this provides a basis for mechanized proofs of locality of curves with Hilbert property (cf. [9]). In addition, this observation allows the identification of seemingly different 3D Hilbert indexings [4], the generalization of a locality result of Gotsman and Lindenbaum [6] to a larger class of multi-dimensional indexing schemes, and the determination that there are exactly $6 \cdot 2^8 = 1536$ structurally different 3D Hilbert curves. The latter clearly generalizes and answers Sagan's quest for describing 3D Hilbert curves [10]. Finally, we provide an easy recursive formula for computing Hilbert indexings in arbitrary dimensions and sketch a recipe for how to construct an r-dimensional Hilbert curve for arbitrary r in an easy way from two $(r-1)$-dimensional ones. Some missing details and proofs can be found in the full version of the paper [1].

2 Preliminaries

We focus our attention on cubic grids, where, grid of side-length n. An r-dimensional *(discrete) curve* C is simply a bijective mapping $C: \{1, \dots n^r\} \to \{1, \dots, n\}^r$. Note that, by definition, we do not claim the continuity of a curve. A curve C is called *continuous* if it forms a Hamilton path through the n^r grid points. An r-dimensional cubic grid is said to be of *order* k if it has side-length 2^k. Analogously a curve C has order k if its range is a cubic grid of order k.

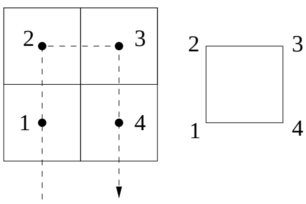


Fig. 1. The generator Hil_1^2 and its canonical corner-indexing $\widetilde{\mathrm{Hil}}_1^2$.

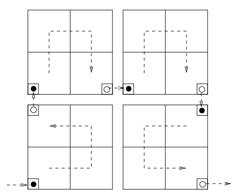


Fig. 2. Construction scheme for the 2D Hilbert indexing.

Fig. 1 shows the smallest 2D continuous curve indexing a grid of size 4. This curve can be found in Hilbert's original work (see [11]) as a constructing unit for a whole family of curves. Fig. 2 shows the general construction principle for these so-called Hilbert curves: For any $k \geq 1$ four Hilbert indexings of size 4^k

are combined into an indexing of size 4^{k+1} by rotating and reflecting them in such a way that concatenating the indexings yields a Hamilton path through the grid. One of the main features of the Hilbert curve is its "self-similarity". Here "self-similar" shall simply mean that the curve can be generated by putting together identical (basic construction) units, only applying rotation and reflection to these units. In a sense, the Hilbert curve is the "simplest" self-similar, recursive, locality-preserving indexing scheme for square meshes of size $2^k \times 2^k$.

3 Formalizing Hilbert curves in r dimensions

In this section, we generalize the construction principle of 2D Hilbert curves to arbitrary dimensions in a rigorous, mathematically precise way.

3.1 Classes of Self-Similar Curves and their generators

Let $V_r := \{x_1x_2\cdots x_{r-1}x_r \mid x_i \in \{0,1\}\}$ be the set of all 2^r corners of an r-dimensional cube coded in binary. Moreover, let $\mathcal{I} : V_r \longrightarrow \{1,\ldots,2^r\}$ denote an arbitrary indexing of these corners. To describe the orientation of subcurves inside a curve of higher order, we want to use symmetry mappings, which can be expressed via suitable permutations operating on such corner-indexings. Observe that any r-dimensional curve C_1 of order 1 naturally induces an indexing of these corners (see Fig. 1 and Fig. 3). We call the obtained corner-indexing the *canonical* one and denote it by $\widetilde{C_1} : V_r \longrightarrow \{1,\ldots,2^r\}$. Furthermore, let $W_{\mathcal{I}}$ denote the group of all permutations (operating on $\mathcal{I}$) that describe rotations and reflections of the r-dimensional cube. In other words, $W_{\mathcal{I}}$ is the set of all permutations that preserve the neighborhood-relations $n(i,j)$ of the corner indexing $\mathcal{I}$:

$$W_{\mathcal{I}} := \{\pi \in Sym(2^r) \;:\; n(i,j) = n(\pi(i),\pi(j)) \;\; \forall i,j \in \{1,\ldots,2^r\}\}.$$

For a given permutation $\tau \in W_{\mathcal{I}}$, we sometimes write $(\tau : \mathcal{I})$ in order to emphasize that τ is operating on a cube with corner-indexing $\mathcal{I}$. The point here is that once we have fixed a certain corner-indexing $\mathcal{I}$, the set $W_{\mathcal{I}}$ will provide all necessary transformations to describe a construction principle of how to generate curves of higher order by piecing together a suitable curve of lower order. Obviously each permutation $(\tau : \mathcal{I})$ acting on a given corner-indexing $\mathcal{I}$ canonically induces a bijective mapping on a cubic grid of order k. Subsequently, we do not distinguish between a permutation and the corresponding mapping on a grid.

We partition an r-dimensional cubic grid of order k into 2^r subcubes of order $k-1$. For each $x_1\cdots x_r \in V_r$ we therefore set

$$p^{(k)}_{(x_1\cdots x_r)} := (x_1\cdot 2^{k-1},\ldots,x_r\cdot 2^{k-1}) \in \{0,\ldots,2^k-1\}\times\ldots\times\{0,\ldots,2^k-1\}$$

to be the "lower-left corner" of such a subcube. Let C_{k-1} be an r-dimensional curve of order $k-1$ ($k \geq 2$). Our goal is to define a "self-similar" curve C_k of order k by putting together 2^r pieces of type C_{k-1}. Let $\mathcal{I} : V_r \longrightarrow \{1,\ldots,2^r\}$ be a corner-indexing. We intend to arrange the 2^r subcurves of type C_{k-1} "along"

$\mathcal{I}$. The position of the i'-th (where $i' \in \{1, \dots, 2^r\}$) subcurve inside C_k can formally be described with the help of the grid-points $p^{(k)}_{(x_1 \cdots x_r)}$. Bearing in mind the classical construction principle for the 2D Hilbert indexing, the orientation of the constructing curve C_{k-1} inside C_k can be expressed by using symmetric transformations (that is reflections and rotations). For any sequence of permutations $\tau_1, \dots, \tau_{2^r} \in W_{\mathcal{I}}$ we therefore define

$$C_k(i) := (\tau_{i'} : \mathcal{I}) \circ C_{k-1}\left(i \bmod (2^{k-1})^r\right) + p^{(k)}_{\mathcal{I}^{-1}(i')}, \tag{1}$$

where $i \in \{1, \dots, (2^k)^r\}$ and $i' = (i-1) \operatorname{div} (2^{k-1})^r + 1$. The geometric intuition behind is that the curve C_k can be partitioned into 2^r components of the form C_{k-1} (reflected or rotated in a suitable way). These subcurves are arranged inside C_k "along" the given corner-indexing $\mathcal{I}$. The orientation of the i'-th subcurve inside C_k is described by the effect of $\tau_{i'}$ operating on $\mathcal{I}$.

Definition 1. Whenever two r-dimensional curves C_{k-1} of order $k-1$ and C_k of order k satisfy equation (1) for a given sequence of permutations $\tau_1, \dots, \tau_{2^r} \in W_{\mathcal{I}}$ (operating on the corner-indexing $\mathcal{I} : V_r \longrightarrow \{1, \dots, 2^r\}$), we will write $C_{k-1} \;\overset{\mathcal{I}}{\underset{(\tau_1,\dots,\tau_{2^r})}{\ll}}\; C_k$ and call C_{k-1} the *constructor* of C_k.

Our final goal is to iterate this process starting with a curve C_1 of order 1. It's only natural and in our opinion "preserves the spirit of Hilbert" to fix the corner-indexing according to the structure of the defining curve C_1. Hence, in this situation we can specify our $\mathcal{I}$ to be the canonical corner-indexing $\widetilde{C_1}$. By successively repeating the construction principle in equation (1) k times, we obtain a curve of order k.

Definition 2. Let $\mathcal{C} = \{\, C_k \;\mid\; k \geq 1 \,\}$ be a family of r-dimensional curves of order k. We call $\mathcal{C}$ a *Class of Self-Similar Curves (CSSC)* if there exists a sequence of permutations $\tau_1, \dots, \tau_{2^r} \in W_{\widetilde{C_1}}$ (operating on the canonical corner-indexing $\widetilde{C_1}$) such that for each curve C_k it holds that

$$C_1 \;\overset{\widetilde{C_1}}{\underset{(\tau_1,\dots,\tau_{2^r})}{\ll}}\; C_2 \;\overset{\widetilde{C_1}}{\underset{(\tau_1,\dots,\tau_{2^r})}{\ll}}\; \cdots \;\overset{\widetilde{C_1}}{\underset{(\tau_1,\dots,\tau_{2^r})}{\ll}}\; C_{k-1} \;\overset{\widetilde{C_1}}{\underset{(\tau_1,\dots,\tau_{2^r})}{\ll}}\; C_k.$$

In this case, C_1 is called the *generator of the CSSC* $\mathcal{C}$ and we define the set $\mathcal{H}\big(\, C_1, (\tau_1, \dots, \tau_{2^r})\,\big) := \{\, C_k \;\mid\; k \geq 1 \,\}$ to be the CSSC generated by C_1 and $\tau_1, \dots, \tau_{2^r}$. A CSSC $\mathcal{C} = \{\, C_k \;\mid\; k \geq 1 \,\}$ is called *Class with Hilbert Property (CHP)* if all curves C_k are continuous.

Note that the CSSC $\mathcal{H}\big(\, C_1, (\tau_1, \dots, \tau_{2^r})\,\big)$ is well-defined, because any CSSC is uniquely determined by its generator C_1 and the choice of the permutations $\tau_1, \dots, \tau_{2^r} \in W_{\widetilde{C_1}}$. Our concept for multi-dimensional CHPs only makes use of the very essential tools which can be found in Hilbert's context (cf. [11]) as rotation and reflection. We deliberately avoid more complicated structures (e.g., the use of different sequences of permutations in each inductive step, or the use of several generators for the constructing principle) in order to maintain conceptual simplicity and ease of construction and analysis. However, the theory which we develop in this paper doesn't necessarily restrict to the continuous case. We end this subsection with an example. One easily checks that the classical 2D Hilbert indexing can be described via $\mathcal{H}\big(\,\mathrm{Hil}_1^2,\ ((2\,4),\mathrm{id},\mathrm{id},(1\,3))\,\big) = \{\,\mathrm{Hil}_k^2 \ \mid\ k \geq 1\,\}$, where the generator Hil_1^2 is given in Fig. 1. As Theorem 2 will show, this is the only CHP of dimension 2 "modulo symmetry."

3.2 Disturbing the generator of a CSSC

In this subsection we analyze the effects of disturbing the generator of a CSSC by a symmetric mapping. We will see that any disturbance of the generator will be hereditary to the whole CSSC in a very canonical way. And also the other way round: if two different CSSCs show a certain similarity in one of their members, this similarity can already be found in the structure of the corresponding generators. We illustrate this by the following diagram. Given two CSSCs $\mathcal{H}\big(\,C_1,\,(\tau_1,\ldots,\tau_{2^r})\,\big) = \{C_k \mid k \geq 1\}$ and $\mathcal{H}\big(\,D_1,\,(\tau_1,\ldots,\tau_{2^r})\,\big) = \{D_k \mid k \geq 1\}$, respectively.[1] Suppose there is a similarity at a certain stage of the construction, i.e., for some k_0 the curves C_{k_0} and D_{k_0} can be obtained from each other by a similarity transformation Φ. The investigations in this section will show that the inner structure of CSSCs are strong enough to yield the same behavior at the stage of any order.

$$\begin{array}{ccccccc}
C_1 & {}_{(\tau_1,\ldots,\tau_{2^r})}\!\overset{\widetilde{C_1}}{\ll} & C_2 & {}_{(\tau_1,\ldots,\tau_{2^r})}\!\overset{\widetilde{C_1}}{\ll} & \cdots & {}_{(\tau_1,\ldots,\tau_{2^r})}\!\overset{\widetilde{C_1}}{\ll} & C_k \\
\big\downarrow \Phi & \circlearrowright & \big\downarrow \Phi & \circlearrowright & \big\downarrow \Phi & \circlearrowright & \big\downarrow \Phi \\
D_1 & {}_{(\tau_1,\ldots,\tau_{2^r})}\!\overset{\widetilde{D_1}}{\ll} & D_2 & {}_{(\tau_1,\ldots,\tau_{2^r})}\!\overset{\widetilde{D_1}}{\ll} & \cdots & {}_{(\tau_1,\ldots,\tau_{2^r})}\!\overset{\widetilde{D_1}}{\ll} & D_k
\end{array}$$

Consequently, for issues like structural behavior, it will be sufficient to analyze the generating elements of a CSSC only, since we find all necessary information encoded here. We start with a simple observation concerning the behavior of the construction principle of Definition 1 under the "symmetric disturbance" of a constructor. We omit the proof.

Lemma 1. *Let C_{k-1} and C_k be curves of order $k-1$ and k, respectively. Suppose C_{k-1} is the constructor of C_k, i.e., $C_{k-1}\ {}_{(\tau_1,\ldots,\tau_{2^r})}\!\overset{\mathcal{I}}{\ll}\ C_k$, for any sequence of permutations $\tau_1,\ldots,\tau_{2^r} \in W_{\mathcal{I}}$ (acting on a given corner-indexing $\mathcal{I}$). Then for arbitrary $\phi \in W_{\mathcal{I}}$ we have*

$$(\phi : \mathcal{I}) \circ C_{k-1}\ {}_{(\tau_1\circ\phi^{-1},\ldots,\tau_{2^r}\circ\phi^{-1})}\!\overset{\mathcal{I}}{\ll}\ C_k .$$

Whereas, by Lemma 1, we investigated the influence of disturbing the constructor, we now, in a second step, analyze how transforming the underlying

[1] Note that the τ's used in the definition of both CSSCs yield completely different automorphisms on the grid. Whereas in the first case they refer to the corner-indexing $\widetilde{C_1}$, in the second case they act on the corner-indexing $\widetilde{D_1}$, given by generator D_1.

corner-indexing influences the construction principle. We will need such a result, since two different CSSCs (by definition) come up with two different corner-indexings, each of which given by the underlying generator. Again we omit the proof.

Lemma 2. *Given the assumptions of Lemma 1 (that is:* $C_{k-1} \overset{\mathcal{I}}{{}_{(\tau_1,\dots,\tau_{2^r})}\ll} C_k$ *for two curves* C_{k-1} *and* C_k *of successive order), then for arbitrary* $\phi \in W_{\mathcal{I}}$ *and the modified corner-indexing* $\mathcal{K} := \phi^{-1} \circ \mathcal{I}$ *with* $\Phi = (\phi : \mathcal{I}) = (\phi : \mathcal{K})$ *we have*[2]

$$C_{k-1} \overset{\mathcal{K}}{{}_{(\tau_1 \circ \phi,\dots,\tau_{2^r}\circ\phi)}\ll} \Phi \circ C_k \,.$$

Lemma 1 and 2 now allow the proof of the main result of this section. For its illustration we refer to the diagram at the beginning of this section. Do also recall the point made in the footnote there.

Theorem 1. *Let* C_1 *be the generator of the CSSC* $\mathcal{H}\big(C_1, (\tau_1, \dots, \tau_{2^r})\big) = \{C_k \mid k \geq 1\}$ *and* D_1 *the generator of the CSSC* $\mathcal{H}\big(\, D_1,\, (\tau_1, \dots, \tau_{2^r})\,\big) = \{D_k \mid k \geq 1\}$*. For an arbitrary permutation* $\phi \in W_{\widetilde{C_1}}$ *and the corresponding symmetric mapping* $\Phi = (\phi : \widetilde{C_1}) = (\phi : \widetilde{D_1})$*, the following statements are equivalent:*

(i) $\Phi \circ C_{k_0} = D_{k_0}$ *for some* $k_0 \geq 1$*.*
(ii) $\Phi \circ C_k = D_k$ *for all* $k \geq 1$*.*

Proof. (ii) ⇒ (i) is trivial. For (i) ⇒ (ii) we first show that statement (ii) is true for the generators C_1 and D_1: If $k_0 > 1$ we can divide the cubic grid of order k_0 into 2^r subgrids of order $k_0 - 1$. By the construction principle for CSSCs, the curves C_{k_0} and D_{k_0} traverse these subgrids "along" the canonical corner-indexings $\widetilde{C_1}$ resp. $\widetilde{D_1}$. Since, by assumption, $\Phi \circ C_{k_0} = D_{k_0}$, the corresponding relation also holds true for the corner-indexings $\widetilde{C_1}$ and $\widetilde{D_1}$, which finally yields the validity of the equation $\Phi \circ C_1 = D_1$, because of the isomorphisms $C_1 \simeq \widetilde{C_1}$ resp. $D_1 \simeq \widetilde{D_1}$. We proceed proving (ii) by induction on k. Assuming that $D_k = \Phi \circ C_k$ we show this relation for $k+1$ by applying Lemma 1 and Lemma 2. Since $\{C_k \mid k \geq 1\}$ is a CSSC, we get

$$C_k \overset{\widetilde{C_1}}{{}_{(\tau_1,\dots,\tau_{2^r})}\ll} C_{k+1} \overset{\text{Lemma 1}}{\Longrightarrow} \underbrace{\Phi \circ C_k}_{=D_k} \overset{\widetilde{C_1}}{{}_{(\tau_1\circ\phi^{-1},\dots,\tau_{2^r}\circ\phi^{-1})}\ll} C_{k+1}$$

$$\overset{\text{Lemma 2}}{\Longrightarrow} D_k \overset{\widetilde{D_1}}{{}_{(\tau_1,\dots,\tau_{2^r})}\ll} \Phi \circ C_{k+1}\ ,$$

where the last relation makes use of $\widetilde{D_1} = \phi^{-1} \circ \widetilde{C_1}$, which we immediately obtain from the given equation $D_1 = \Phi \circ C_1$.[3] This implies $D_{k+1} = \Phi \circ C_{k+1}$ because of the CSSC-property of $\{D_k \mid k \geq 1\}$. □

In particular, the result of Theorem 1 implies that any questions concerning the structural similarity of two CSSCs can be reduced to the analysis of their generators.

[2] The fact that the corner-indexing is disturbed by ϕ^{-1} instead of ϕ is due to technical reasons only.

[3] A disturbance by Φ implies a transformation of the corner-indexings by ϕ^{-1}, which can be easily checked.

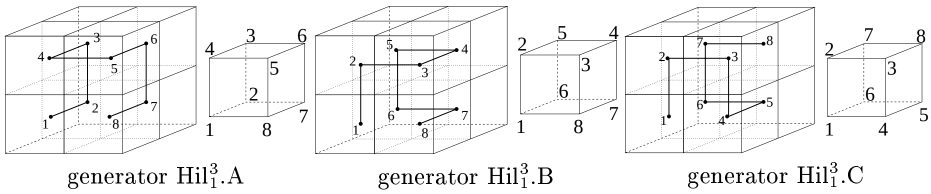


Fig. 3. Continuous 3D generators Hil_1^3.x and their canonical corner-indexings $\widetilde{\mathrm{Hil}_1^3}$.x.

4 Applications: Computing and analyzing CHPs

First in this section, we attack a classification of all structurally different CHPs for higher dimensions. Whereas we can provide concrete combinatorial results for the 2D and 3D cases, the high-dimensional cases appear to be much more difficult. The basic tool for such an analysis, however, is given by Theorem 1. The following theorem justifies the naming "class with Hilbert property" (CHP).

Theorem 2. *The classical 2D Hilbert indexing* $\mathcal{H}\big(\,Hil_1^2,\ ((2\,4), id, id, (1\,3))\,\big)$ *is the only CHP of dimension 2 modulo symmetry.*

Proof. Due to Theorem 1 it suffices to show that Hil_1^2 is the only continuous 2D generator, which is obvious. In addition, we have to check whether there is another sequence of permutations such that 4 generators Hil_1^2 can be arranged in a grid of order 2 along the canonical corner-indexing $\widetilde{\mathrm{Hil}_1^2}$ in a continuous way. A simple combinatorial consideration shows that no other sequence of permutations yields a continuous curve of order 2 whose starting- and endpoints are located at corners of the grid. However, any constructor for a continuous curve of higher order must have this property. □

What about the 3D case? The analysis of the "Simple Indexing Schemes" (which are related to our CHPs) in Chochia and Cole [4] already shows that the number of CHPs in the 3D case grows drastically compared to the 2D setting. However, by our analysis, lots of "Simple Indexing Schemes" in [4] now turn out to be identical modulo symmetry. We state the following classification-theorem, which treats the 3D case entirely. It also generalizes and answers work of Sagan [10].

Theorem 3. *For the 3D case there are* $6 \cdot 2^8 = 1536$ *structurally different (that is: not identical modulo reflection and rotation) CHPs. These types are listed in Table 1.*

Proof (Sketch). Theorem 1 says that we can restrict our attention to checking any continuous curves of order 1 which are different modulo symmetry. Given such a continuous generator C, the total amount of CHPs which can be constructed by C is given by all possibilities of piecing together 8 (rotated or reflected) versions of C ("subcurves") along its canonical corner-indexing $\widetilde{C}$. By exhaustive search, we get that there are 3 different (modulo symmetry) types

Table 1. Description of all 3-dimensional CHPS.

generator	version	τ_1	τ_2	τ_3	τ_4
Hil^3_1.A	(a)	(2 8)(3 5) / (2 4 8)(3 5 7)	(3 7)(4 8) / (2 8 4)(3 7 5)	(3 7)(4 8) / (2 8 4)(3 7 5)	(1 3)(6 8) / (1 3)(2 4)(5 7)(6 8)
	(b)	(2 8)(3 5) / (2 4 8)(3 5 7)	(3 7)(4 8) / (2 8 4)(3 7 5)	id / (2 4)(5 7)	(1 7 3)(4 6 8) / (1 7 5 3)(2 8 6 4)
	(c)	(2 8)(3 5) / (2 4 8)(3 5 7)	(3 7)(4 8) / (2 8 4)(3 7 5)	id / (2 4)(5 7)	(1 7)(4 6) / (1 7 5)(2 6 4)
	(d)	(2 8)(3 5) / (2 4 8)(3 5 7)	(3 7)(4 8) / (2 8 4)(3 7 5)	(3 7)(4 8) / (2 8 4)(3 7 5)	(1 3)(6 8) / (1 3)(2 4)(5 7)(6 8)
Hil^3_1.B	(a)	(2 8)(5 7) / (2 6 8)(3 5 7)	id / (2 6)(3 7)	(3 5)(6 8) / (2 8 6)(3 7 5)	(2 8)(5 7) / (2 6 8)(3 5 7)
	(b)	(2 8)(5 7) / (2 6 8)(3 5 7)	id / (2 6)(3 7)	(3 5)(6 8) / (2 8 6)(3 7 5)	(3 5)(6 8) / (2 8 6)(3 7 5)
generator	**version**	τ_5	τ_6	τ_7	τ_8
Hil^3_1.A	(a)	(1 3)(6 8) / (1 3)(2 4)(5 7)(6 8)	(1 5)(2 6) / (1 5 7)(2 4 6)	(1 5)(2 6) / (1 5 7)(2 4 6)	(1 7)(4 6) / (1 7 5)(2 6 4)
	(b)	(1 3 5)(2 6 8) / (1 3 5 7)(2 4 6 8)	id / (2 4)(5 7)	(1 5)(2 6) / (1 5 7)(2 4 6)	(1 7)(4 6) / (1 7 5)(2 6 4)
	(c)	(2 8)(3 5) / (2 4 8)(3 5 7)	id / (2 4)(5 7)	(1 5)(2 6) / (1 5 7)(2 4 6)	(1 7)(4 6) / (1 7 5)(2 6 4)
	(d)	(1 3 5)(2 6 8) / (1 3 5 7)(2 4 6 8)	id / (2 4)(5 7)	(1 5)(2 6) / (1 5 7)(2 4 6)	(1 7)(4 6) / (1 7 5)(2 6 4)
Hil^3_1.B	(a)	(1 3)(4 6) / (1 3 7)(2 4 6)	(1 3)(4 6) / (1 3 7)(2 4 6)	id / (2 6)(3 7)	(1 7)(2 4) / (1 7 3)(2 6 4)
	(b)	(1 7)(2 4) / (1 7 3)(2 6 4)	(1 3)(4 6) / (1 3 7)(2 4 6)	id / (2 6)(3 7)	(1 7)(2 4) / (1 7 3)(2 6 4)

of continuous generators, namely Hil^3_1.A, Hil^3_1.B and Hil^3_1.C (see Fig. 3). As described above, we now have to check whether there are continuous arrangements of these generators along their canonical corner-indexings. Beginning with type A, an exhaustive combinatorial search yields that there are 4 possible continuous formations of Hil^3_1.A along $\widetilde{\text{Hil}^3_1}$.A. All possibilities are shown in Fig. 4, where the orientation of each subcube is given by the position of an edge (drawn in bold lines). For each subcube there are two symmetry mappings which yield possible arrangements for the generator within such a subgrid. The permutations expressing these mappings are listed in Table 1. Analogously, we find out the possible arrangements for generator type B. Note that there are no more than 2 different continuous arrangements of this generator along its canonical corner-indexing. Finally we easily check that Hil^3_1.C cannot even be the constructor of a continuous curve of order 2. Table 1 thus yields that there are exactly $4 \cdot 2^8 + 2 \cdot 2^8 = 6 \cdot 2^8$ structurally different CHPs. □

Construction of an r-dimensional Hilbert curve. Without giving an explicit proof here, we just indicate how the construction of a high-dimensional CHP can be done inductively: A continuous generator of dimension r can be derived simply by "joining together" two continuous generators of dimension $r-1$. As an example we give a CHP of dimension 4, whose generator Hil^4_1 is constructed by joining together two generators Hil^3_1, version (a) (cf. Figure 3). The generator Hil^4_1 and a suitable sequence of permutations are shown in Fig. 5. Note that this construction principle can be extended to obtain Hilbert indexings in arbitrary dimensions in an expressive and easy, constructive way: Following the

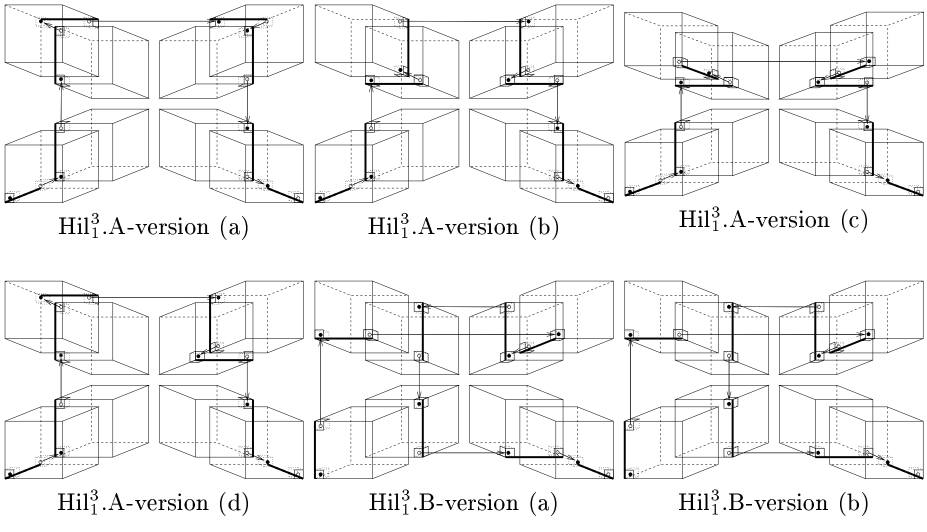
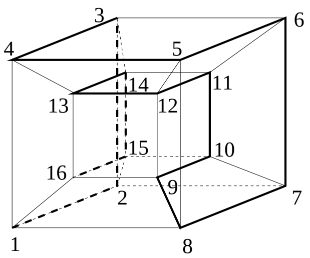


Fig. 4. Construction principles for CHPs with generators Hil^3_1.A and Hil^3_1.B.

construction principle of Hil^3_1, version (a), first pass through an $r-1$-dimensional structure, then in "two steps" do a change of dimension in the rth dimension, and finally again pass through an $r-1$-dimensional structure. This method applies to finding the generators as well as to finding the permutations.

Recursive computation of CSSCs. Note that whenever a CSSC $\mathcal{C} = \{C_k \mid k \geq 1\}$ is explicitly given by its generator and the sequence of permutations, we may use the recursive formula (1) of Subsection 3.1 to compute the curves C_k. In other words, the defining formula (1) itself provides a computation-scheme for CSSC, which is parameterized by the generating elements (generator and sequence of permutations).

Aspects of locality. The above mentioned parameterized formula might, for example, also be used to investigate locality properties of CSSCs by mechanical methods. The locality properties of Hilbert curves have already been studied in great detail. As an example for such investigations, we briefly note a result of Gotsman and Lindenbaum [6] for multi-dimensional Hilbert curves. In [6] they investigate a curve $C:\ \{1,\dots,n^r\} \to \{1,\dots,n\}^r$ with the help of their locality measure $L_2(C) = \max_{i,j\in\{1,\dots,n^r\}}(d_2(C(i),C(j)))/|i-j|$, where d_2 denotes the Euclidean metric. In their Theorem 3 they claim the upper bound $L_1(H^r_k) \leq (r+3)^{\frac{r}{2}}2^r$ for any r-dimensional Hilbert curve of order k, without precisely specifying what an r-dimensional Hilbert curve shall be. Since the proof of their result does not utilize the special Hilbert structure of the curve, this result can even be extended to arbitrary CSSCs.



$$
\begin{aligned}
\tau_1 &= (2\ 16)(3\ 13)(6\ 12)(7\ 9)\\
\tau_2 &= (3\ 15)(4\ 16)(5\ 9)(6\ 10)\\
\tau_3 &= \tau_2\\
\tau_4 &= (1\ 3\ 13\ 11\ 9\ 7)(2\ 4\ 14\ 12\ 10\ 8)(5\ 15)(6\ 16)\\
\tau_5 &= \tau_4\\
\tau_6 &= (1\ 5\ 13\ 9)(2\ 6\ 14\ 10)(3\ 11\ 15\ 7)(4\ 12\ 16\ 8)\\
\tau_7 &= \tau_6\\
\tau_8 &= (1\ 7)(4\ 6)(10\ 16)(11\ 13)\\
\tau_9 &= \tau_8\\
\tau_{10} &= (1\ 9\ 13\ 5)(2\ 10\ 14\ 6)(3\ 7\ 15\ 11)(4\ 8\ 16\ 12)\\
\tau_{11} &= \tau_{10}\\
\tau_{12} &= (1\ 11)(2\ 12)(3\ 5\ 7\ 9\ 15\ 13)(4\ 6\ 8\ 10\ 16\ 14)\\
\tau_{13} &= \tau_{12}\\
\tau_{14} &= (1\ 13)(2\ 14)(7\ 11)(8\ 12)\\
\tau_{15} &= \tau_{14}\\
\tau_{16} &= (1\ 15)(4\ 14)(5\ 11)(8\ 10)
\end{aligned}
$$

Fig. 5. Constructing elements for a 4-D CHP (generator Hil_1^4 and permutations).

5 Conclusion

Our paper lays the basis for several further research directions. So it could be tempting to determine the number of structurally different r-dimensional curves with Hilbert property for $r > 3$. Moreover, a (mechanized) analysis of locality properties of r-dimensional ($r > 3$) Hilbert curves is still to be done (cf. [9]). An analysis of the construction of more complicated curves using more generators or different permutations for different levels remains open.

References

1. J. Alber and R. Niedermeier. On multi-dimensional Hilbert indexings. Technical Report 98-392, KAM-DIMATIA Series, Faculty of Mathematics and Physics, Charles University, Praha, Czech Republic, May 1998.
2. A. R. Butz. Space filling curves and mathematical programming. *Information and Control*, 12:314–330, 1968.
3. A. R. Butz. Convergence with Hilbert's space filling curve. *Journal of Computer and System Sciences*, 3:128–146, 1969.
4. G. Chochia and M. Cole. Recursive 3D mesh indexing with improved locality. Technical report, University of Edinburgh, 1997. Short version appeared in the Proceedings of HPCN'97, LNCS 1225.
5. G. Chochia, M. Cole, and T. Heywood. Implementing the Hierarchical PRAM on the 2D mesh: Analyses and experiments. In *Symposium on Parallel and Distributed Processing*, pages 587–595. IEEE Computer Science Press, October 1995.
6. C. Gotsman and M. Lindenbaum. On the metric properties of discrete space-filling curves. *IEEE Transactions on Image Processing*, 5(5):794–797, May 1996.
7. H. V. Jagadish. Linear clustering of objects with multiple attributes. *ACM SIGMOD Record*, 19(2):332–342, June 1990.
8. I. Kamel and C. Faloutsos. Hilbert R-tree: An improved R-tree using fractals. In *20th International Conference on Very Large Data Bases*, pages 500–509, 1994.
9. R. Niedermeier, K. Reinhardt, and P. Sanders. Towards optimal locality in mesh-indexings. In *Proceedings of the 11th Conference on Fundamentals of Computation Theory*, number 1279 in Lecture Notes in Computer Science, pages 364–375, Kraków, Poland, Sept. 1997. Springer-Verlag.
10. H. Sagan. A three-dimensional Hilbert curve. *Int. J. Math. Educ. Sci. Technol.*, 24(4):541–545, 1993.
11. H. Sagan. *Space-Filling Curves.* Universitext. Springer-Verlag, 1994.