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Refined Search Tree Technique for Dominating Set on Planar Graphs

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Abstract. We establish refined search tree techniques for the parameterized DOMINATING SET problem on planar graphs. We derive a fixed parameter algorithm with running time $O(8^k n)$, where k is the size of the dominating set and n is the number of vertices in the graph. For our search tree, we firstly provide a set of reduction rules. Secondly, we prove an intricate branching theorem based on the Euler formula. In addition, we give an example graph showing that the bound of the branching theorem is optimal with respect to our reduction rules. Our final algorithm is very easy (to implement); its analysis, however, is involved.

Keywords. dominating set, planar graph, fixed parameter algorithm, search tree

1 Introduction

The parameterized DOMINATING SET problem, where we are given a graph $G = (V, E)$, a parameter k and ask for a set of vertices of size at most k that dominate all other vertices, is known to be $W[2]$-complete for general graphs [8]. The class $W[2]$ formalizes intractability from the point of view of parameterized complexity. It is well-known that the problem restricted to planar graphs is fixed parameter tractable. An algorithm running in time $O(11^k n)$ was claimed in [7, 8]. The analysis of the algorithm, however, turned out to be flawed; hence, this paper seems to give the first completely correct analysis of a fixed parameter algorithm for DOMINATING SET on planar graphs with running time $O(c^k n)$ for *small* constant c that even improves the previously claimed constant considerably. We mention in passing that in companion work various approaches that yield algorithms of running time $O(c^{\sqrt{k}} n)$ for PLANAR DOMINATING SET and related problems were considered (see [1–3]).[1] Interestingly, very recently it was

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[1] The huge worst case constant c that was derived there is rather of theoretical interest.

shown that up to the constant c this time bound is optimal unless a very unlikely collapse of a parameterized complexity hierarchy occurs [4, 5].

Fixed parameter algorithms based on search trees. A method that has proven to yield easy and powerful fixed parameter algorithms is that of constructing a bounded search tree. Suppose we are given a graph class $\mathcal{G}$ that is closed under taking subgraphs and that guarantees a vertex of degree d for some constant d.[2] Such graph classes are, e.g., given by bounded degree graphs, or by graphs of bounded genus, and, hence, in particular, by planar graphs. More precisely, an easy computation shows that, e.g., the class $\mathcal{G}(S_g)$ of graphs that are embeddable on an orientable surface S_g of genus g guarantees a vertex of degree $d_g := \lceil 2(1+\sqrt{3g+1})\rceil$ for $g > 0$, and $d_0 := 5$.

Consider the k-INDEPENDENT SET problem on $\mathcal{G}$, where, for given $G = (V, E) \in \mathcal{G}$, we seek for an independent set of size at least k. For a vertex u with degree at most d and neighbors $N(u) := \{u_1, \ldots, u_d\}$, we can choose one vertex $w \in N[u] := \{u, u_1, \ldots, u_d\}$ to be in an optimal independent set and continue the search on the graph G' where we deleted $N[w]$. This observation yields a simple $O((d+1)^k n)$ degree-branching search tree algorithm.

In the case of k-DOMINATING SET, the situation seems more intricate. Clearly, again, either u or one of its neighbors can be chosen to be in an optimal dominating set. However, removing u from the graph leaves all its neighbors being already dominated, but still also being suitable candidates for an optimal dominating set. This consideration leads us to formulate our search tree procedure in a more general setting, where there are two kinds of vertices in our graph.

ANNOTATED DOMINATING SET
Input: A *black* and *white* graph $G = (B \uplus W, E)$, and a positive integer k.
Parameter: k
Question: Is there a choice of at most k vertices $V' \subseteq V = B \uplus W$ such that, for every vertex $u \in B$, there is a vertex $u' \in N[u] \cap V'$? In other words, is there a set of k vertices (which may be either black or white) that dominates the set of all black vertices?

In each step of the search tree, we would like to branch according to a low degree black vertex. By our assumptions on the graph class, we can guarantee the existence of a vertex $u \in B \uplus W$ with $\deg(u) \leq d$. However, as long as *not all* vertices have degree bounded by d (as, e.g., the case for graphs of bounded genus g, where only *the existence* of a vertex of degree at most d_g is known), this vertex need not necessarily be black. These considerations show that a direct $O((d+1)^k n)$ search tree algorithm for DOMINATING SET seems out of reach for such graph classes.

Our results. In this paper, we present a fixed parameter algorithm for (ANNOTATED) DOMINATING SET on planar graphs with running time $O(8^k n)$. For that purpose, we provide a set of reduction rules and, then, use a search tree in which we are constantly simplifying the instance according to the reduction rules (see

[2] This means that, for each $G = (V, E) \in \mathcal{G}$, there exists a $u \in V$ with $\deg_G(u) \leq d$.

Subsection 3.1). The branching in the search tree will be done with respect to low degree vertices. The analysis of this algorithm will be carried out in a new branching theorem (see Subsection 3.2) which is based on the Euler formula for planar graphs. In addition, we give an example showing that the bound of the branching theorem is optimal (see Section 3.3). Finally, it is worth noting here that the algorithm we present is very simple and easy to implement.

Due to the lack of space, several proof details had to be omitted.

2 Preliminaries

We assume familiarity with basic notions and concepts in graph theory, see, e.g., [6]. For a graph $G = (V, E)$ and a vertex $u \in V$, we use $N(u)$ and $N[u]$, respectively, to denote the open and closed neighborhood of u, respectively. By $\deg_G(u) := |N_G(u)|$, we denote the *degree* of the vertex u in G. A *pendant* vertex is a vertex of degree one. For $V' \subseteq V$, the induced subgraph of V' is denoted by $G[V']$. In particular, we use the abbreviation $G - V' := G[V \setminus V']$. If V' is a singleton, then we omit brackets and simply write $G - v$ for a vertex v. In addition, we write $G - e$ or $G + e$ when we delete or add an edge e to G without changing the vertex set of G.

Let G be a connected planar graph, i.e., a connected graph that admits a crossing-free embedding in the plane. Such an embedding is called a *plane embedding*. A planar graph together with a plane embedding is called a *plane graph*. Note that a plane graph can be seen as a subset of the Euclidean plane $\mathbb{R}^2$. The set $\mathbb{R}^2 \setminus G$ is open; its regions are the *faces* of G. Let $\mathcal{F}$ be the set of faces of a plane graph. The *size of a face* $F \in \mathcal{F}$ is the number of vertices on the boundary of the face. A *triangular face* is a face of size three. If G is a plane graph and $V' \subseteq V$, then $G[V']$ and $G - V'$ can be always considered as plane graphs with an embedding inherited by the embedding of G.

3 The algorithm and its analysis

Our algorithm is based on reduction rules (see Subsection 3.1) and an improved branching theorem (see Subsection 3.2). With respect to our set of reduction rules, we show optimality for the branching theorem (see Subsection 3.3).

3.1 Reduction rules

We consider the following reduction rules for simplifying the ANNOTATED PLANAR DOMINATING SET problem. In developing the search tree, we will always assume that we are branching from a reduced instance (thus, we are constantly simplifying the instance according to the reduction rules).[3] When a vertex u is

[3] The idea of doing so-called *rekernelizations* (i.e., repeated application of reduction rules) while constructing the search tree was already exhibited in [9, 10] in a somewhat different context.

placed in the dominating set D by a reduction rule, then the target size k for D is reduced to $k-1$ and the neighbors of u are whitened.

(R1) Delete edges between white vertices.
(R2) Delete a pendant white vertex.
(R3) If there is a pendant black vertex w with neighbor u (either black or white), then delete w, place u in the dominating set, and lower k to $k-1$.
(R4) If there is a white vertex u of degree 2, with two black neighbors u_1 and u_2 connected by an edge $\{u_1, u_2\}$, then delete u.
(R5) If there is a white vertex u of degree 2, with black neighbors u_1, u_3, and there is a black vertex u_2 and edges $\{u_1, u_2\}$ and $\{u_2, u_3\}$ in G, then delete u.
(R6) If there is a white vertex u of degree 2, with black neighbors u_1, u_3, and there is a white vertex u_2 and edges $\{u_1, u_2\}$ and $\{u_2, u_3\}$ in G, then delete u.
(R7) If there is a white vertex u of degree 3, with black neighbors u_1, u_2, u_3 for which the edges $\{u_1, u_2\}$ and $\{u_2, u_3\}$ are present in G (and possibly also $\{u_1, u_3\}$), then delete u.

Let us call a set of simplifying reduction rules of a certain problem *sound* if, whenever (G, k) is some problem instance and instance (G', k') is obtained from (G, k) by applying one of the reduction rules, then (G, k) has a solution iff (G', k') has a solution. A simple case analysis shows:

Lemma 3.1. *The reduction rules are sound.* □

Suppose that G is a *reduced* graph, that is, none of the above reduction rules can be applied. By using the rules (R1), (R2), (R4) and (R7), we can show:

Lemma 3.2. *Let $G = (B \uplus W, E)$ be a plane black and white graph. If G is reduced, then the white vertices form an independent set and every triangular face of $G[B]$ is empty.* □

3.2 A new branching theorem

Theorem 3.3. *If $G = (B \uplus W, E)$ is a planar black and white graph that is reduced, then there exists a black vertex $u \in B$ with $\deg_G(u) \leq 7$.*

The following technical lemma, based on an "Euler argument," will be needed. Note that if there is any counterexample to the theorem, then there is a connected counterexample.

Lemma 3.4. *Suppose $G = (B \uplus W, E)$ is a connected plane black and white graph with b black vertices, w white vertices, and e edges. Let the subgraph induced by the black vertices be denoted $H = G[B]$. Let c_H denote the number of components of H and let f_H denote the number of faces of H. Let*

$$z = \big(3(b + w) - 6\big) - e \tag{1}$$

measure the extent to which G fails to be a triangulation of the plane. If the criterion

$$3w - 4b - z + f_H - c_H < 7 \tag{2}$$

is satisfied, then there exists a black vertex $u \in B$ with $\deg_G(u) \leq 7$.

Proof. Let the (total) numbers of vertices, edges and faces of G be denoted v, e, f respectively. Let e_{bw} be the number of edges in G between black and white, and let e_{bb} denote the number of edges between black and black. With this notation, we have the following relationships.

$$v - e + f = 2 \qquad \text{(Euler formula for } G\text{)} \tag{3}$$

$$v = b + w \tag{4}$$

$$e = e_{bb} + e_{bw} \tag{5}$$

$$b - e_{bb} + f_H = 1 + c_H \qquad \text{((extended) Euler formula for } H\text{)} \tag{6}$$

$$2v - 4 - z = f \qquad \text{(by Eq. (1), (4), and (5))} \tag{7}$$

If the lemma were false, then we would have, using (5),

$$8b \;\leq\; 2e_{bb} + e_{bw} \;=\; e_{bb} + e. \tag{8}$$

We will assume this and derive a contradiction. The following inequality holds:

$$\begin{aligned} 3 + c_H &= v + b - (e_{bb} + e) + f + f_H && \text{(by (3) and (6))} \\ &\leq v + b - 8b + f + f_H && \text{(by (8))} \\ &= 3v - 7b + f_H - 4 - z && \text{(by (7))} \\ &= 3w - 4b + f_H - 4 - z. && \text{(by (4))} \end{aligned}$$

This yields a contradiction to 2. □

Proving Theorem 3.3 by contradiction, it will be helpful to know that a corresponding graph has to be connected and has minimum degree 3.

Lemma 3.5. *If there is any counterexample to Theorem 3.3, then there is a connected counterexample where $\deg_G(u) \geq 3$ for all $u \in W$.*

Proof. Suppose G is a counterexample to the theorem. Then, G does not have any white vertices of degree 1, else reduction rule (R2) can be applied. Let G' be obtained from G by simultaneously replacing every white vertex u of degree 2 with neighbors x and y by an edge $\{x, y\}$. The neighbors x and y of u are necessarily black, else (R1) can be applied, and in each case the edge $\{x, y\}$ is not already present in G, else rule (R4) would apply. We argue that G' is reduced. If not, then the only possibility is that reduction rule (R7) applies to some white vertex u of degree 3 in G'. If rule (R7) did not apply to u in G, then one of the edges between the neighbors of u must have been created in our derivation of G' from G, i.e., one of these edges replaced a white vertex u' of degree 2. But this implies that reduction rule (R6) could be applied in G to u', contradicting that G is reduced. □

Before giving the proof of Theorem 3.3, we introduce the following notation:

Notation: Let $G = (B \uplus W, E)$ be a plane black and white graph and let $\mathcal{F}$ be the set of faces of $G[B]$. Then, for each $F \in \mathcal{F}$, we let

- w_F denote the number of white vertices embedded in F,
- z_F denote the number of edges that would have to be added in order to complete a triangulation of that part of the embedding of G contained in F,
- t_F denote the number of edges needed to triangulate F in $G[B]$ (that is, triangulating only between the black vertices on the boundary of F, and noting that the boundary of F may not be connected), and
- c_F denote the number of components of the boundary of F, minus 1.

Observe that the numbers z_F and t_F are indeed well-defined. This can be shown by using [6, Proposition 4.2.6].

Proof (of Theorem 3.3). We can assume that if there is a counterexample G then G is connected, but the black subgraph $H := G[B]$ might not be connected. Moreover, by Lemma 3.5 we may assume that $\deg_G(u) \geq 3$ for all $u \in W$. If c_H denotes the number of components of H, by induction on c_H, it is easy to see that $c_H - 1 = \sum_{F \in \mathcal{F}} c_F$. Also, if z is the number of edges needed to triangulate G, we clearly get $z = \sum_{F \in \mathcal{F}} z_F$. The criterion (2) established by Lemma 3.4 can be rephrased as

$$3 \sum_{F \in \mathcal{F}} w_F - \sum_{F \in \mathcal{F}} z_F \ - 4b + f_H - c_H \ < \ 7,$$

which is equivalent to

$$3 \sum_{F \in \mathcal{F}} (w_F + c_F/3 - z_F/3 + 1/3) \ - 4b - 2c_H \ < \ 6.$$

Now, assume that we can show the inequality

$$w_F + c_F/3 - z_F/3 + 1/3 \ \leq \ \alpha t_F + \beta \tag{9}$$

for some constants α and β and for every face F of the subgraph H. Call this our *linear bound* assumption. Then, criterion (2) will hold if

$$3 \sum_{F \in \mathcal{F}} (\alpha t_F + \beta) - 4b - 2c_H = \left(3\alpha \sum_{F \in \mathcal{F}} t_F \right) + \left(3\beta \sum_{F \in \mathcal{F}} 1 \right) - 4b - 2c_H \ < \ 6.$$

Noting that $\sum_{F \in \mathcal{F}} t_F$ is the number of edges needed to triangulate H, we have

$$\sum_{F \in \mathcal{F}} t_F = 3b - 6 - e_{bb}.$$

The number of faces of H is $\sum_{F \in \mathcal{F}} 1 = f_H = e_{bb} - b + 1 + c_H$, by Euler's formula (7). Together, these give us the following targeted criterion:

$$3\alpha(3b - 6 - e_{bb}) + 3\beta(e_{bb} - b + 1 + c_H) - 4b - 2c_H < 6.$$

Multiplying out and gathering terms, we need to establish (using the linear bound assumption), that

$$b(9\alpha - 3\beta - 4) + e_{bb}(3\beta - 3\alpha) + 3\beta(1 + c_H) - 18\alpha - 2c_H < 6.$$

This inequality is easily verified for $\alpha = \beta = 2/3$.

To complete the argument, we need to establish that the linear bound assumption (9) with $\alpha = \beta = 2/3$ holds for faces of reduced graphs, i.e., that

$$w_F + c_F/3 - z_F/3 \leq 2t_F/3 + 1/3. \tag{10}$$

But this is a consequence of the following Propositions 3.6 and 3.8. □

Proposition 3.6. *Let $G = (B \uplus W, E)$ be a reduced plane black and white graph and let F be a face of $G[B]$. Then, using the notation above, we have $w_F + c_F \leq z_F + 1$.*

Proof. Consider the "face-graph" $G_F := G[B_F \cup W_F]$, where B_F is the set of black vertices forming the boundary of F and W_F is the set of white vertices inside F. Note that G_F may consist of several "black components," connected among themselves through white vertices. Contracting each of these black components into one (black) vertex, we obtain the *bipartite* graph G'_F. Note that both the black and also the white vertices form independent sets in G'_F. Clearly, G'_F is still planar. Since G'_F is a bipartite planar graph, preserving planarity it is easy to show that we can connect the white vertices among themselves by a tree of $w_F - 1$ white white edges and that we can connect the black vertices among themselves by a tree of c_F black black edges. Clearly, this implies that we can also add at least $c_F + w_F - 1$ new edges to G_F without destroying planarity. Hence, we need at least $c_F + w_F - 1$ additional edges to triangulate the interior of F in the graph G. □

Property 3.7. If F_1 and F_2 are two faces of $G[B]$ with common boundary edge e, then $t_{F_1} + t_{F_2} + 1$ equals t_F, where we now consider $(G - e)[B]$, and F is the face which results from merging F_1 and F_2 when deleting e.

Proposition 3.8. *Suppose $G = (B \uplus W, E)$ is a reduced plane black and white graph, with $\deg(u) \geq 3$ for all $u \in W$. Let F be a face of $G[B]$. Then, using the notation above, $w_F \leq t_F$.*

Proof. Consider a reduced black and white graph $G = (B \uplus W, E)$ with $\deg(u) \geq 3$ for all $u \in W$. If there is some $u \in W$ with $\deg(u) > 4$, then delete arbitrarily all edges incident with u but four of them. While preserving the black induced subgraph, the resulting graph is still reduced, since no rules apply to white degree-4-vertices. Therefore, we can assume from now on without loss of generality that all white vertices of G have maximum degree of four.

We will now show the claim by induction on the number $\#w^4$ of white vertices of degree four. The hardest part is the induction base, which is deferred to the subsequent Lemma 3.9. Assume that the claim was shown for each graph with

$\#w^4 \leq \ell$ and assume now that G has $\ell+1$ white degree-4-vertices. Choose some arbitrary $u \in W$ with $\deg(u) = 4$. Let $\{b_1, \ldots, b_4\}$ be the clockwisely ordered neighbors of u. Due to planarity, we may assume further that $\{b_1, b_3\} \notin E$ without loss of generality. Consider now $G' = (G-u)+\{b_1, b_3\}$. We prove below that G' (or $G'' = (G-u) + \{b_2, b_4\}$ in one special case) is reduced. This means that the induction hypothesis applies to G'. Hence, $w_F \leq t_F$ for all faces in $G'[B]$. Observe that G' contains all the faces of G except from the face F of G which contains u; F might be replaced by two faces F_1 and F_2 with common boundary edge $\{b_1, b_3\}$. In this case, $w_{F_1} \leq t_{F_1}$, $w_{F_2} \leq t_{F_2}$, $w_{F_1} + w_{F_2} + 1 = w_F$ and, by Property 3.7, $t_{F_1} + t_{F_2} + 1 = t_F$. Hence, $w_F \leq t_F$ by induction. In the case where face F still exists in G' it is trivial to see that $w_F \leq t_F$.

To complete the proof, we argue why G' has to be reduced. Obviously, this is clear if $\forall b_i \forall v \in N(b_i) \deg(v) = 4$, since no reduction rules apply to degree-4-vertices. We now discuss the case that u has degree-3-vertices as neighbors.

1. If a degree-3-vertex is neighbor of some b_i, but not of b_j, $j \neq i$, then no reduction rule is triggered when constructing G'.
2. Consider the case that a degree-3-vertex is neighbor of two b_i, b_j, $i \neq j$. We can assume that $\{i, j\} \neq \{1, 3\}$, since otherwise $\{b_2, b_4\} \notin E$ and we could consider $G'' = (G - u) + \{b_2, b_4\}$ instead of G' with a argument similar to the case $\{i, j\} = \{1, 3\}$. If $\{i, j\} = \{1, 3\}$, then G' is clearly reduced. If $\{i, j\} = \{1, 2\}$ (or, more generally, $|\{i, j\} \cap \{1, 3\} = 1|$), then no reduction rules are triggered when passing from G to G'.
3. If a degree-3-vertex is neighbor of three b_i, b_j, b_k, then a reasoning similar to the one in the previous point applies.

This concludes the proof of the proposition. □

The following lemma serves as the induction base in the proof of Proposition 3.8.

Lemma 3.9. *Suppose $G = (B \uplus W, E)$ is a reduced plane black and white graph, with $\deg(u) = 3$ for all $u \in W$. Let F be a face of $G[B]$. Then, using the notation above, $w_F \leq t_F$.*

Proof. (Sketch) Let us consider a fixed planar embedding of the graph G, and consider a face F of the black induced subgraph $G[B]$. Let $W_F \subseteq W$ be the set of white vertices in the interior of F, and let $B_F \subseteq B$ denote the black vertices on the boundary of F. We want to find at least $|W_F|$ many black black edges that can be added to $G[B]$ inside F. For that purpose, define the set

$$E^{\text{poss}} := \{e = \{b_1, b_2\} \mid b_1, b_2 \in B_F \ \wedge \ e \notin E(G[B])\}$$

of non-existing black black edges.

For a subset $W' \subseteq W_F$ we construct a bipartite graph $H(W') := (W' \cup T(W'), E(W'))$ as follows. In $H(W')$, the first bipartition set is formed by the vertices W' and the second one is given by the set

$$T(W') := \{e = \{b_1, b_2\} \in E^{\text{poss}} \mid \exists u \in W' : e \subset N_G(u)\}.$$

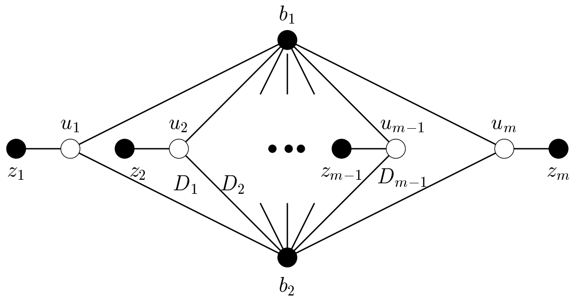

Fig. 1. Illustration of a diamond D generated by a pair vertices $\{b_1, b_2\} \in T(W_F)$.

The edges in $H(W')$ are then given by

$$E(W') := \big\{\{u, e\} \mid u \in W', e \in T(W'), e \subset N_G(u)\big\}.$$

In this way, the set $T(W')$ gives us vertices in $H(W')$ that correspond to pairs $e = \{b_1, b_2\}$ of black vertices in B_F between which we still can draw an edge in $G[B]$. Note that the edge e can even be drawn in the interior of F, since b_1 and b_2 are connected by a white vertex in $W' \subseteq W_F$. This means that

$$|T(W_F)| \leq t_F. \tag{11}$$

Due to reduction rule (R7), for each $u \in W_F$, the neighbors $N(u) \subseteq B_F$ are connected by at most one edge in $G[B]$. By construction of $H(W_F)$, we find:

$$\deg_{H(W_F)}(u) \geq 2 \quad \forall u \in W_F. \tag{12}$$

The degree $\deg_{H(W_F)}(e)$ for an element $e = \{b_1, b_2\} \in T(W_F)$ tells us how many white vertices share the pair $\{b_1, b_2\}$ as common neighbors. We do case analysis according to this degree.

Case 1: Suppose $\deg_{H(W_F)}(e) \leq 2$ for all $e \in T(W_F)$, then $H(W_F)$ is a bipartite graph, in which the first bipartition set has degree at least two (see Eq. (12)) and the second bipartition set has degree at most two. In this way, the second set cannot be smaller, which yields

$$w_F \;=\; |W_F| \;\leq\; |T(W_F)| \;\stackrel{(11)}{\leq}\; t_F.$$

Case 2: There exist elements $e = \{b_1, b_2\}$ in $T(W_F)$ which are shared as common neighbors by more than 2 white vertices (i.e., $\deg_{H(W_F)}(e) = m > 2$). Suppose we have $u_1, \ldots, u_m \in W_F$ with $N_G(u_i) = \{b_1, b_2, z_i\}$ (i.e., $\{u_i, e\} \in E(W_F)$). We may assume that the vertices are ordered such that the closed region D bounded by $\{b_1, u_1, b_2, u_m\}$ contains all other vertices $u_2, \ldots, u_{m-1}$ (see Figure 1).

We call D the *diamond* generated by $\{b_1, b_2\}$. Note that D consists of $m - 1$ regions, which we call *blocks* in the following; the block D_i is bounded by

$\{b_1, u_i, b_2, u_{i+1}\}$ $(i = 1, \ldots, m-1)$. Let $W_i \subseteq W_F$, and $B_i \subseteq B_F$, respectively, denote the white and black, respectively, vertices that lie in D_i. For the boundary vertices $\{b_1, b_2, u_1, \ldots, u_m\}$ we use the following convention: b_1, b_2 are added to all blocks, i.e., $b_1, b_2 \in B_i$ for all i; and u_i is added to the region where its third neighbor z_i lies in. A block is called *empty* if $B_i = \{b_1, b_2\}$ and, hence, $W_i = \emptyset$. Moreover, let $W_D := \bigcup_{i=1}^{m-1} W_i$ and $B_D := \bigcup_{i=1}^{m-1} B_i$.

We only consider diamonds, where z_1 and z_m are not contained in D (see Figure 1). The other cases can be treated with similar arguments.

Note that each block of a diamond D may contain further diamonds, the blocks of which may contain further diamonds, and so on. Since no diamonds overlap, the topological inclusion forms a natural ordering on the set of diamonds and their blocks.

We can show the following claim by induction on the diamond structure:

Claim: For each diamond D generated by $\{b_1, b_2\}$, we can add t_D (where $t_D \geq |W_D|$) many black black edges to $G[B]$ other than $\{b_1, b_2\}$. All of these additional edges can be drawn inside D so that $\{b_1, b_2\}$ still can be drawn.

Using this claim, we can finish the proof of the induction base of the proposition: Consider all diamonds $D^1, \ldots, D^r$ which are not contained in any further diamond. Suppose D^i has boundary $\{b_1^i, u_1^i, b_2^i, u_{m_i}^i\}$ with $b_1^i, b_2^i \in B_F$ and $u_1^i, u_{m_i}^i \in W_F$. Let

$$W'_F := W_F \setminus (\bigcup_{i=1}^{r} W_{D^i}).$$

According to the claim we already found $\sum_{i=1}^{r} t_{D^i}$ many black black edges in E^{poss} inside the diamonds D^i. Observe that each pair $e^i = \{b_1^i, b_2^i\}$ is only shared as common neighbors by at most two white vertices (namely, u_1^i and $u_{m_i}^i$) in (sic!) W'_F. Hence, the bipartite graph $H(W'_F)$ again has the property that

- $\deg_{H(W'_F)}(e) \leq 2$ for all $e \in T(W'_F)$ and still
- $\deg_{H(W'_F)}(u) \geq 2$ for all $u \in W'_F$.[4]

Similar to "Case 1" this proves that—additionally—we find t' (with $t' \geq |W'_F|$) many edges in E^{poss}. Hence,

$$w_F \;=\; |W_F| \;=|W'_F| + \left|\bigcup_{i=1}^{r} W_{D^i}\right| \;\leq\; t' + (\sum_{i=1}^{r} t_{D^i}) \;\leq\; t_F. \qquad \square$$

Using Theorem 3.3 for the construction of a search tree as elaborated in Section 1, we conclude:

Theorem 3.10. (Annotated) dominating set *on planar graphs can be solved in time $O(8^k n)$.* □

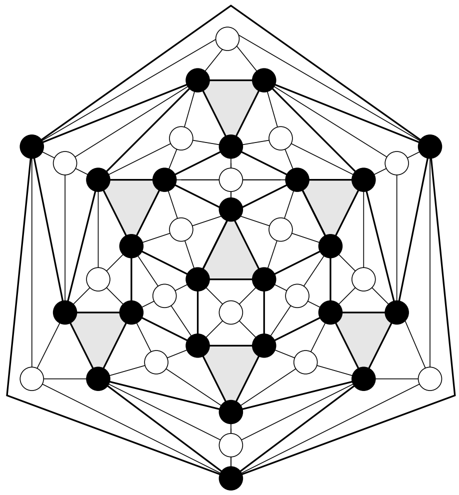

Fig. 2. A reduced graph with all black vertices having degree 7, thus showing the optimality of the bound derived in our branching theorem.

3.3 Optimality of the branching theorem

We conclude this section by the observation that with respect to the set of reduction rules introduced above the upper bound in our branching theorem is optimal. More precisely, there exists a plane reduced black and white graph with the property that all black vertices have degree at least 7. Such a graph is shown in Figure 2. Moreover, this example can be generalized towards an infinite set of plane graphs with the property that all black vertices have degree at least 7. The given example is the smallest of all graphs in this class. It is an interesting and challenging task to ask for further reduction rules that would yield a provably better constant in the branching theorem. For example, one might think of the following generalization of reduction rule (R6):

(R6') If there are white vertices $u_1, u_2 \in W$ with $N_G(u_1) \subseteq N_G(u_2)$, then delete u_1.

However, the graph in Figure 2 is reduced even with respect to rule (R6'). We leave it as an open question to come up with further reduction rules such that the graph of Figure 2 is no longer reduced.

4 Conclusion and open questions

In this paper, we gave the first search tree algorithm proven to be correct for the DOMINATING SET problem on planar graphs. It improves on the original,

[4] Note that according to the claim the edges $\{b_1^i, b_2^i\}$ still can be used.

flawed theorem stating an exponential term 11^k, which is now lowered to 8^k. Unfortunately, the proof of correctness has become considerably more involved and fairly technical.

Our work suggests several directions for future research:

- Can we improve the branching theorem by adding further, more involved reduction rules?
- Finding a so-called problem kernel (see [8] for details) of polynomial size $p(k)$ for DOMINATING SET on planar graphs in time $T_K(n, k)$ would improve the running time to $O(8^k + T_K(n, k))$ using the interleaving technique analyzed in [10]. Currently, we even hope for a linear size problem kernel for DOMINATING SET on planar graphs.
- Since our results for the search tree itself are based on the Euler formula, a generalization to the class of graphs $\mathcal{G}(S_g)$ (allowing a crossing-free embedding on an orientable surface S_g of genus g) seems likely.

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